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A Reformulation of the Feynman Chessboard Model

G. N. Ord[1]

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The chessboard model is reviewed and reformulated as a four-state process. In this formulation both the Dirac propagator of the chessboard model and the partition function of the associated Ising chain are observed to be projections of a single matrix of partition functions onto two orthogonal eigenspaces. This helps clarify the role played by the phase associated with Feynman paths in this model.

KEY WORDS: Feynman chessboard model; quantum stochastics; Ising chains.

1. INTRODUCTION

The one-dimensional Ising model is a very useful "prototype" system in statistical physics. On one hand, the system is sufficiently simple that it permits a number of exact solutions under the right circumstances.[1-3] On the other hand, the physics is sufficiently rich that it provides valuable insight into a wide range of problems.[4-7]

One of the more intriguing aspects of the one-dimensional Ising model is its connection with the Feynman chessboard model (FCM) first pointed out by Gersch.[8] The model itself was originally proposed by Feynman[9] as an illustration of the path integral approach to quantum mechanics in a relativistic context.

In Section 2 we shall sketch Gersch's derivation to establish the connection between the quantum problem and classical statistical mechanics.

In Section 3 we pursue the quantum–classical analogy further by reformulating the FCM as a four-state process. In this formulation both the

[1] Department of Applied Mathematics, University of Western Ontario, London, Ontario N6A 5B9, Canada.

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quantum and classical problems are seen as projections of the same problem onto two different eigenspaces. This makes the "statistical roots" of the FCM rather more transparent than in previous formulations.

2. THE FEYNMAN CHESSBOARD MODEL

Feynman and Hibbs[4] proposed a relativistic model for a particle moving in one spatial dimension. In their model the particle was constrained to move with speed $c=1$ on a space-time lattice with lattice spacing ε. The kernel $K(b, a)$ for a particle to propagate from position a at time t_a to position b at time t_b was given by

$$K(b, a)=\sum_R N(R)(i\varepsilon m)^R \tag{2.1}$$

where the sum is over all forward "bishop's moves" connecting the space-time points. The set of such paths (Fig. 1) is subdivided into paths with R corners, and any path with R corners is given the weight $(i\varepsilon m)^R$. Here $N(R)$ is the number of paths with R corners. In the limit as $\varepsilon \to 0$, provided $(b-a)/(t_b-t_a) \ll 1$, the above sum over paths approaches the correct Dirac free-particle kernel.

The connection between the above sum and the partition function of a one-dimensional Ising model has been established by Gersch,[8] with a more detailed account by Jacobson and Schulman.[10] Here we shall briefly sketch the arguments involved, and refer the interested reader to the above two articles for more detailed treatments.

Referring to Fig. 1, we can see that the Feynman paths in this model are constructed from only two kinds of elementary "links." These links may

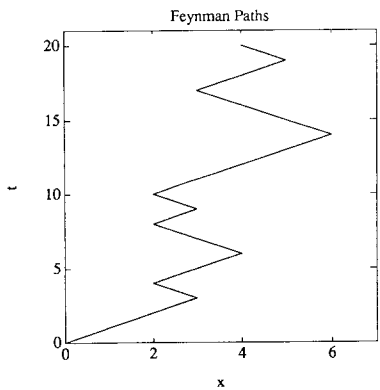

Fig. 1. Typical path in the Feynman sum.

be labeled as + or − according to their displacement in the x direction. The + links correspond to a particle moving in the positive x direction for a time ε, and the − links correspond to a particle moving in the negative x direction for a time ε. Note that at any lattice point on the t axis, t_i say, the displacement of the path $x(t_i) \equiv x_i$ is simply $x_i = x_{i-1} + \varepsilon\sigma_{i-1}$, where $\sigma_{i-1} = \pm 1$ is an Ising spin variable. This suggests that we may represent any N-step Feynman path by a set of N spins $\{\sigma_1, \sigma_2, ..., \sigma_N\}$. Each path will correspond to a unique configuration of these N spins, and the displacement $x(t_i)$ of the path will be ε times the displacement of the ith spin in magnetization space. The "corners" in the Feynman paths correspond to adjacent antiparallel spins (or domain boundaries) in the spin chain. Since the Feynman sum (2.1) weights such corners, we notice that in the spin chain description the function $\frac{1}{2}(1 - \sigma_i\sigma_{i+1})$ is 1 if σ_i and σ_{i+1} are antiparallel and 0 if they are parallel. This function conveniently counts the domain boundaries in the spin chain when summed on i. Finally, we note that the "corner weight" in (2.1) is imaginary. This corresponds to the Feynman "phase rule," which associates a quantum phase change of $\pi/2$ for each corner in the path.

In order to maintain the Ising analogy, we shall replace the corner weight of $(i\varepsilon m)$ in Eq. (2.1) with the real positive weight (εm), in which case the kernel (2.1) is replaced by a partition function of an Ising chain. (We may later recover the FCM by replacing m by im.) That is, we now consider

$$\begin{aligned} \mathscr{Z}_{\sigma_N\sigma_1}(X, N, \varepsilon) &= \sum_R N(R)(\varepsilon m)^R \\ &= \sum_{\sigma_2 = \pm 1} \sum_{\sigma_3 = \pm 1} \cdots \sum_{\sigma_{N-1}} (\varepsilon m)^{(\sum_{i=1}^{N-1}(1 - \sigma_i\sigma_{i+1}))/2} \end{aligned} \tag{2.2}$$

where the sums over $\sigma_2, ..., \sigma_{N-1}$ include only those configurations for which the magnetization is fixed at X (i.e., $\sum_{i=1}^{N} \sigma_i = X$). Here we have omitted the sum over the initial and final spins σ_1 and σ_N, so that $\mathscr{Z}_{\sigma_N\sigma_1}(X, N, \varepsilon)$ is a matrix of partition functions whose elements are labeled by the spins at the beginning and end of the chain. We shall now refer to $\mathscr{Z}$ as a partition function, since its matrix character is unimportant in the Ising context.

The sum in (2.2) is still awkward to calculate because of the constraint of fixed magnetization. To avoid this constraint, we consider the lattice Fourier transform of $\mathscr{Z}$ in magnetization space, i.e., define

$$Z_{\sigma_N\sigma_1}(p, N, \varepsilon) = \sum_{X=-N}^{N} e^{-iXp\varepsilon} \mathscr{Z}(X, N, \varepsilon) \tag{2.3}$$

Now, if we define the nearest neighbor interaction strength (in units of kT) as

$$j = -\tfrac{1}{2}\ln(\varepsilon m) \tag{2.4}$$

we can write (2.3) as

$$Z_{\sigma_N\sigma_1}(p, N, \varepsilon) = \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_{N-1} = \pm 1} \times \exp\left(-ip\varepsilon \sum_{n=1}^{N} \sigma_n + j \sum_{n=1}^{N-1} (\sigma_n \sigma_{n+1} - 1) \right) \tag{2.5}$$

Here the Ising sum is unconstrained and (2.5) may be evaluated using a transfer matrix. Before we perform the evaluation, we consider the role of ε in (2.5). From the analogy between the spin chain and the kernel (2.1) we are interested in the limit as $\varepsilon \to 0$. In this limit j goes to infinity (or the temperature goes to zero), so that domain boundaries become less and less frequent. If we fix N at any finite value as $\varepsilon \to 0$, we shall simply recover the uninteresting ferromagnetic ground state of the one-dimensional Ising model. If, however, in the spirit of the kernel (2.1), we consider a sequence of lattice spacings ε in which $t \equiv N\varepsilon$ is fixed, the limit $\varepsilon \to 0$ is more interesting. In this limit the probability per spin of finding a domain boundary is proportional to $e^{-2j} = \varepsilon m$. However, the number of spins per "unit chain length" (or per unit time in the FCM picture) is $1/\varepsilon$. Thus, the expected domain size (or time between corners) is proportional to $1/m$. In the Ising system the limit we are contemplating is one in which the size of the system and the temperature are related in such a way that the expected number of domains is fixed as $T \to 0$.

With this in mind, we evaluate (2.5). Consider the 2×2 transfer matrix

$$T(\sigma, \sigma') = \exp\left[-\frac{i}{2} p\varepsilon(\sigma + \sigma') + j(\sigma\sigma' - 1) \right] \tag{2.6}$$

In terms of this matrix the sum (2.5) is

$$Z_{\sigma_N\sigma_1}(p, N, \varepsilon) = e^{-ip\varepsilon(\sigma_1 + \sigma_N)/2} (T^{N-1})_{\sigma_N\sigma_1} \tag{2.7}$$

This is a standard result which simply recognizes that the summations in Eq. (2.5) correspond exactly to the summations involved in the matrix product T^{N-1}. However, since we shall later want to avoid writing summations like (2.5) explicitly, it is worthwhile interpreting the transfer matrix directly. Explicitly, the transfer matrix T is

$$T = \begin{pmatrix} e^{-ip\varepsilon} & e^{-2j} \\ e^{-2j} & e^{ip\varepsilon} \end{pmatrix} = \begin{pmatrix} e^{-ip\varepsilon} & \varepsilon m \\ \varepsilon m & e^{ip\varepsilon} \end{pmatrix} \tag{2.8}$$

where we have used Eq. (2.4) for the off-diagonal Boltzmann factors. To see that this matrix does "count" weighted configurations of the spin chain, note that the diagonal entries in the matrix simply correspond to "transitions" from one spin to a spin of the same sign. This raises or lowers the magnetization by one unit in the "counting variable" $-\varepsilon p$. The off-diagonal terms correspond to transitions from one spin to another spin of the opposite sign. This contributes a domain boundary weight of εm, but does not change the magnetization. For example, if we consider T^2, we have

$$T^2 = \begin{pmatrix} e^{-2ip\varepsilon} + \varepsilon^2 m^2 & \varepsilon m(e^{ip\varepsilon} + e^{-ip\varepsilon}) \\ \varepsilon m(e^{ip\varepsilon} + e^{-ip\varepsilon}) & e^{+2ip\varepsilon} + \varepsilon^2 m^2 \end{pmatrix} \tag{2.9}$$

The (1, 1) element of T^2 simply counts the weighted configurations of chains of length 3 which begin and end with $\sigma_1 = \sigma_3 = +1$. The term $e^{-2ip\varepsilon}$ corresponds to the configuration $\sigma_1 = \sigma_2 = \sigma_3 = +1$ and the two added spins σ_2 and σ_3 contribute the magnetization $2p\varepsilon$. The term $(\varepsilon m)^2$ corresponds to the spin triple $(+1, -1, +1)$, which has two boundaries, and the added spins σ_2 and σ_3 contribute nothing to the magnetization. Similarly, the (1, 2) element of T^2 corresponds to the spin triples $(-1, +1, +1)$, and $(-1, -1, +1)$, respectively.

In general, then, the (σ_N, σ_1) element of T^N will be a polynomial in the exponential $(e^{i\varepsilon p})$. The coefficient of $e^{-iM\varepsilon p}$ in this polynomial will be a power series in the variable (εm), i.e.,

$$(T^N)_{\sigma_N \sigma_1} = \sum_{M=-N}^{N} e^{-iMp\varepsilon} \left(\sum_{R=0}^{N} C(M, R)(\varepsilon m)^R \right) \tag{2.10}$$

Here $C(M, R)$ is the number of configurations with R domain boundaries such that $\sum_{n=2}^{N} \sigma_n = M$. The sum in brackets is the fixed magnetization partition function $\mathscr{Z}$ of Eq. (2.2).

The point of making explicit this calculation is then twofold. First of all let us note that finding the term in brackets in Eq. (2.10) by inverting the discrete Fourier transform is considerably easier than evaluating the constrained sum in Eq. (2.2) directly.

The second pertinent observation is that the transfer matrix T itself is sufficiently simple that if we wish to generalize this problem in any way, it may well be easier to write down the transfer matrix directly without having to construct an explicit spin formulation. This fact will be used in the next section.

We now return to the evaluation of the transformed partition function $Z(p, N, \varepsilon)$, Eq. (2.7). We write

$$T = \lambda_+ \mathbf{P}^+ + \lambda_- \mathbf{P}^- \tag{2.11}$$

where $\lambda_\pm$ are the eigenvalues of T with corresponding orthogonal projectors $\mathbf{P}^\pm$. We then have from (2.7), for small ε,

$$Z(p, N, \varepsilon) = (\lambda_+^{N-1}\mathbf{P}^+ + \lambda_-^{N-1}\mathbf{P}^-) \tag{2.12}$$

and the partition function problem is reduced to finding the eigenvalues and eigenvectors of T. Since we are interested in the limit as $\varepsilon \to 0$, we expand T to first order in ε and find that

$$T = \begin{pmatrix} 1 - i\varepsilon p & \varepsilon m \\ \varepsilon m & 1 + i\varepsilon p \end{pmatrix} + O(\varepsilon^2) \tag{2.13}$$

and to lowest order in ε

$$\lambda_\pm = 1 \pm \varepsilon(m^2 - p^2)^{1/2} \tag{2.14}$$

Both eigenvalues are real and positive for small ε ($p^2 > m^2$ corresponds to the unphysical situation of $M > N$) and as usual the larger eigenvalue λ_+ will dominate.

If we now fix the chain length at $t = N\varepsilon$ and consider the limit as $\varepsilon \to 0$, we find that

$$(\lambda^+)^{N-1} \to [1 + \varepsilon(m^2 - p^2)^{1/2}]^{t/\varepsilon} \to e^{(m^2-p^2)^{1/2}t} \tag{2.15}$$

Thus, in this limit the partition function (2.12) becomes

$$Z(p, t) = e^{(m^2-p^2)^{1/2}t}\mathbf{P}^+ \tag{2.16}$$

In the limit $p^2 \ll m^2$ corresponding to chains with small endpoint magnetization this is

$$Z(p, t) \simeq e^{mt}e^{-(1/2m)p^2t}\mathbf{P}^+ \tag{2.17}$$

The form of Eq. (2.17) suggests that since Z is Gaussian in p, the chain end magnetization will also be Gaussian as appropriate to a "diffusive" process. We note that this "diffusive" character of the magnetization is a consequence of the reality of the eigenvalues of T in Eq. (2.14). Physically, this is what we expect, since in zero magnetic field the chain magnetization simply executes a random walk in magnetization space.

Thus far we have considered only the Ising problem with real weights. However, to consider the FCM with imaginary weights, we simply return to Eq. (2.12), and in the subsequent discussion replace m by im everywhere. Thus for the FCM the transfer matrix is

$$T_F = \begin{pmatrix} 1 - i\varepsilon p & i\varepsilon m \\ i\varepsilon m & 1 + i\varepsilon p \end{pmatrix} + O(\varepsilon^2) \tag{2.18}$$

The eigenvalues are

$$\lambda_{\pm} = 1 \pm i\varepsilon E \tag{2.19}$$

with

$$E = (m^2 + p^2)^{1/2} \tag{2.20}$$

Note that here the eigenvalue are complex and of equal modulus. Both eigenvalues will contribute to the propagator in (2.12). Repeating the arguments preceding Eq. (2.16), one finds that the expression analogous to (2.16) is

$$K(p,t) = e^{iEt}\mathbf{P}_F^{+} + e^{-iEt}\mathbf{P}_F^{-} \tag{2.21}$$

where K is now a propagator for the Feynman problem and $\mathbf{P}_F^{\pm}$ are the projection operators corresponding to the two eigenvalues in (2.19). Calculating $\mathbf{P}_F^{\pm}$ explicitly, one finds[8]

$$K_{11}(p,t) = \tfrac{1}{2}(1 + p/E)\, e^{-iEt} + \tfrac{1}{2}(1 - p/E)\, e^{iEt} \tag{2.22}$$

$$K_{22}(p,t) = \tfrac{1}{2}(1 - p/E)\, e^{-iEt} + \tfrac{1}{2}(1 + p/E)\, e^{iEt} \tag{2.23}$$

$$K_{12}(p,t) = -(1/2E)\, e^{-iEt} + (1/2E)\, e^{iEt}$$

$$= K_{21}(p,t) \tag{2.24}$$

The kernels $K(z,t)$ may be obtained from the above by inverting the Fourier transform, i.e.,

$$K_{11}(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, e^{ipx} K_{11}(p,t) \tag{2.25}$$

The momentum space kernels (2.22)–(2.24) are the same as those obtained from the Dirac equation.[10]

Now the difference between the partition function of the Ising problem and the kernel of the Feynman problem lies primarily in the characteristics of the eigenvalues of the transfer matrix. (For a detailed discussion of this see the article by Jacobson and Schulman.[10])

The complex exponentials arising in the propagator (2.22)–(2.24) are a direct result of the complex eigenvalues in Eq. (2.19). These in turn are a consequence of the imaginary "corner weights" $i\varepsilon m$ in the off-diagonal elements of the transfer matrix in (2.18). However, as soon as we use imaginary boundary weights, we lose any direct interpretation through classical statistical mechanics. This raises the question, "Is there a formulation of the FCM in which the transfer matrix has real 'corner weights,' and

the propagator arises directly from a 'classical' partition function calculation?" The answer to this question is "yes," as we shall see in the following section.

3. The Reformulation

In the previous section the sum over paths, Eq. (2.1), could equally well be written for finite ε as

$$K(b, a) = \sum_{R=0,4,8,\ldots} N(R)(\varepsilon m)^R - \sum_{R=2,6,10,\ldots} N(R)(\varepsilon m)^R + i\left(\sum_{R=1,5,\ldots} N(R)(\varepsilon m)^R - \sum_{R=3,7,\ldots} N(R)(\varepsilon m)^R \right) \tag{3.1}$$

Note that here, each separate sum over paths has a real weight. In this form a clear distinction can be made between the classical partition function of the spin problem and the quantum propagator. Any classical partition function represents a *sum* over independent configurations. From a statistical point of view the minus signs in (3.1) allow the configurations of the ensemble to interfere with each other. They give rise to the "sum over *interfering* alternatives" of the path integral formulation.

However, each separate sum in (3.1) is simply a classical sum, and we can formulate the problem in a way that allows us to calculate each sum separately. The reason for doing this is that it allows us to keep the "statistical" problem completely in the realm of classical statistical mechanics, where interpretation is simple. In this way we hope to make the role of phase in the quantum problem more transparent.

In accordance with (3.1), we see that there are actually four states in the quantum problem. For example, the spin sequences or paths $(+, -, +, -, +)$ and $(+, +, -, -, +)$ have the same magnetization (displacement), but since they differ by two boundaries (corners), they differ in sign in their contribution to the Feynman sum. We can account for this if we suppose that the end spins of the two chains are in two different "states." In the previous formulation the two states were uniquely defined by the direction of motion. In this formulation we will have to duplicate both states. For example, if we imagine building a sequence of spins and asking at each step, "which sum in Eq. (3.1) would this sequence contribute to if it terminated here?" we would find that the same pattern is always repeated. That is, the sequence would contribute to the first sum until the first reversed spin was added. It would then contribute to the third sum until the second domain boundary was encountered, at which point it would contribute to the second sum. At the next reversal the contribution would switch to the fourth sum, and finally after four boundaries the con-

tribution would revert back to the first sum. The repeating sequence is $1 \to 3 \to 2 \to 4 \to 1$ in the above sums.

We now consider assigning four possible states to a given spin, depending on its contribution to the Feynman sum (3.1). States 1 and 2, which are identical in the Ising system, but which differ by a minus sign in the Feynman sum, both corresponding to a right step or spin $+1$. States 3 and 4 both correspond to a left step or spin -1. Now consider the transfer matrix

$$T_4 = \begin{pmatrix} e^{-ip\varepsilon} & 0 & 0 & \varepsilon m \\ 0 & e^{-ip\varepsilon} & \varepsilon m & 0 \\ \varepsilon m & 0 & e^{+ip\varepsilon} & 0 \\ 0 & \varepsilon m & 0 & e^{+ip\varepsilon} \end{pmatrix} \tag{3.2}$$

Looking at the diagonal elements, $1 \to 1$ and $2 \to 2$ transitions both increase the magnetization "counter" $-p\varepsilon$ and $3 \to 3$ and $4 \to 4$ transitions decrease the magnetization, as they should. Looking at each column in turn, $1 \to 3$, $2 \to 4$, $3 \to 2$, and $4 \to 1$ are the only state changes allowed, reproducing the pattern $1 \to 3 \to 2 \to 4$ required by the Feynman sum (3.1). T_4 will simply "count contributions" to the desired sum, as did T and T_F in the previous section. Equations (2.11) and (2.12) have simple extensions to the 4×4 system and the calculation is straightforward. We now have only to consider "contraction" back to the 2×2 versions of the previous section.

Now in the Ising problem there is no distinction between states 1 and 2 or betwen states 3 and 4. Furthermore, there is no "phase" distinction between the first two and last two "states." In terms of the sums in (3.1), this means that in the Ising equivalent there is no i and no minus signs. To contract to the 2×2 representation for the Ising system, we simply have to merge the states 1 and 2, as well as 3 and 4. That is, we regard states 1 and 3 as being the "real" states of the Ising system. We "start" all configurations in states 1 and 3, and we merge equivalent states at the end of the calculation.

For example, a matrix element from state 1 to state 2 in the 2×2 Ising system will correspond to the sum of two matrix elements in the 4×4 representation, i.e., $1 \to 3 + 1 \to 4$. To accomplish the "contraction" through matrix multiplication, consider the "starting contractor"

$$c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.3}$$

and the "merging contractor"

$$c^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{3.4}$$

If $A = [a_{ij}]$ is a general 4×4 matrix, then

$$c^* A c = \begin{pmatrix} a_{11} + a_{21} & a_{13} + a_{23} \\ a_{31} + a_{41} & a_{33} + a_{43} \end{pmatrix} \tag{3.5}$$

Thus, left and right multiplication by c^* and c, respectively, provides the desired contraction to the 2×2 Ising representation. In particular, we note that

$$c^* T_4 c = \begin{pmatrix} e^{ip\varepsilon} & \varepsilon m \\ \varepsilon m & e^{+ip\varepsilon} \end{pmatrix} \tag{3.6}$$

This is precisely the transfer matrix of the Ising chain (2.8).

Similar considerations for the Feynman problem suggest the quantum contractors

$$q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -i \\ 0 & 0 \end{pmatrix} \tag{3.7}$$

and

$$q^* = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & +i & -i \end{pmatrix} \tag{3.8}$$

We find that

$$q^* T_4 q = \begin{pmatrix} e^{-ip\varepsilon} & i\varepsilon m \\ i\varepsilon m & e^{+ip\varepsilon} \end{pmatrix} \tag{3.9}$$

This is the transfer matrix (2.18) for the Feynman problem.

We are now in a position to formulate both the Ising and Feynman problems in terms of T_4. The Ising partition function is

$$\begin{aligned} Z(p, t) &= c^* [\lim_{\varepsilon \to 0} T^{t/\varepsilon}] c \\ &= c^* [\lim_{\varepsilon \to 0} (\lambda_1^{t/\varepsilon} \mathbf{P}_1 + \lambda_2^{t/\varepsilon} \mathbf{P}_2 + \lambda_3^{t/\varepsilon} \mathbf{P}_3 + \lambda_4^{t/\varepsilon} \mathbf{P}_4)] c \end{aligned} \tag{3.10}$$

and the Feynman kernel is similarly

$$K(p, t) = q^*[\lim_{\varepsilon \to 0} T_4^{t/\varepsilon}]q \tag{3.11}$$

where here the λ_K and $\mathbf{P}_K$ are the eigenvalues and corresponding projection operators of T_4. Notice that the statistical problem of counting configurations is now the same for both the Ising and Feynman cases. We choose to "input" quantum phase differences (or not) at the contraction stage by choosing the contractors (q^*, q) [or (c^*, c)].

To see how this works, we calculate the eigenvalues and projectors of T_4. The eigenvalues are found to be

$$\lambda^c_{\pm} = 1 \pm i\varepsilon(m^2 + p^2)^{1/2} \equiv 1 \pm i\varepsilon E \tag{3.12}$$

and

$$\lambda^r_{\pm} = 1 \pm \varepsilon(m^2 - p^2)^{1/2} \equiv 1 \pm \varepsilon F \tag{3.13}$$

The complex eigenvalues (3.12) are precisely those of the Feynman problem (2.19), and the real eigenvalues (3.13) are those of the Ising problem (2.14). The orthogonal projection operators of (3.10) are readily computed. For example, the projectors for the eigenvalue $\lambda^c_{\pm}$ are

$$P^c_{\pm} = \begin{pmatrix} (1 \mp p/E)\,\Sigma^- & +(im/E)\,\Sigma^- \\ -(im/E)\,\Sigma^- & (1 \pm p/E)\,\Sigma^- \end{pmatrix} \tag{3.14}$$

with

$$\Sigma^- = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{3.15}$$

The projectors for the real eigenvalues are

$$P^r_{\pm} = \begin{pmatrix} (1 \mp ip/F)\Sigma & (m/F)\Sigma \\ (m/F)\Sigma & (1 \pm ip/F)\Sigma \end{pmatrix} \tag{3.16}$$

with

$$\Sigma = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.17}$$

A quick check shows that the contractors q^* and q "select" the complex eigenvalues. That is,

$$q^* P^r_{\pm} q = 0$$

and the real eigenvalues which control the spin problem are removed from the kernel calculation (3.11). We are then left with

$$K(p, t) = \lim_{\varepsilon \to 0} q^*(\lambda_+^{ct/\varepsilon} P_+^c + \lambda_-^{ct/\varepsilon} P_-^c) q \tag{3.18}$$

This reduces to the 2×2 system of equations (2.22)–(2.24).

Similarly, if we replace the quantum contractors q and q^* with the classical contractors c and c^*, we find that $c^* P_{\pm}^c c = 0$ and the real eigenvalues are selected. If we then find the partition function of Eq. (3.10), this reduces to

$$Z(p, t) = c^* \lim_{\varepsilon \to 0} [(\lambda_+^{rt/\varepsilon} \mathbf{P}_+^r + \lambda_-^{rt/\varepsilon} \mathbf{P}_-^r] c \tag{3.19}$$

This in turn reduces to Eq. (2.16).

To summarize, we have now reformulated both the Ising and chessboard problems so that their relationship may be closely inspected. In the 4×4 formulation the transfer matrix T_4 is a completely "classical" object in that it calculates the Fourier transforms of classical partition functions. [Note that the presence of complex exponentials in T_4, (3.2), reflects only the counting involved in the Fourier transform. The i in these exponentials has nothing to do with any quantum mechanical phase.] In spite of this, T_4 contains all the information necessary to form both the Feynman propagator and the Ising partition function. Both of these quantities are obtained by projection of T_4^N onto two separate eigenspaces.

On one hand, the fact that T_4 contains all the information necessary to solve both problems is not surprising, since we designed T_4 to calculate separately the Fourier transforms of each of the sums in Eq. (3.1). On the other hand, previous formulations of the FCM related the "quantum mechanical" nature of the propagator directly to the presence of imaginary corner weights in the transfer matrix, and consequently to the resulting complex eigenvalues. These imaginary corner weights removed the calculation from the domain of classical statistical mechanics. This raises the question, "How does the classical object T_4 'know' about the quantum calculation and the corresponding complex eigenvalues?" Upon inspection of T_4, one can see that the answer to this is simply that it is the periodic structure of T_4 which generates the complex eigenvalues, and the "quantum propagator" is obtained by simply exploiting this periodicity in the "right" way. Ignoring the periodicity and merging the appropriate states gave us the Ising partition function. Fully distinguishing the four states with the phases of Eq. (3.1) gave us the Dirac kernel. However, in both cases the calculation of the path statistics is the same.

4. CONCLUSIONS

The Feynman chessboard model has recently received attention in a classical context as a description of semiflexible polymers.[11] Like the underlying one-dimensional Ising model, the FCM has the virtue of being a simple system which nonetheless provides a graphic illustration of some complex physics.

The new formulation in which a four-state transfer matrix is the central object displays the relation between the statistical geometry of the paths and the phases of the paths in a particularly transparent way. The statistics of both the quantum and classical versions of the problem are obtained from the same eigenvalue calculation. The propagator and partition function are both extracted as projections onto different eigenspaces.

The end result is that a link is established between the Feynman phase rule on one hand, and an eigenspace of a *classical* walk on the other. In a subsequent publication we shall use this link to show that the Feynman phase rule itself has a direct ancestor within classical physics. This ancestor is the "classical antiparticle" of Wheeler and Feynman[12] and Stuckelberg.[11]

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